Announcements

1) HW #4 due Thursday

2) Quiz next week

Properties of Laplace transform, cont.

3) We showed

$$\mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)$$

In general, if $f^{(k)}(t)$ denotes the $k^{\underline{th}}$ derivative of f,

$$\boxed{\mathcal{L}(f^{(n)})(s) = s^n\mathcal{L}(f)(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(s)}$$

4) $\mathcal{L}\{t^n f(t)\}(s)$

$$= (-1)^n \frac{d^n F}{ds^n}(s)$$

Unfortunate, since now we don't just convert differential equations to algebraic equations, we get another differential equation when applying the Laplace Transform to equations with nonconstant coefficients.

Moral: The Laplace transform is best confined to constant coefficient differential equations.

Periodic Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic if there is a real number $T > 0$ such that $\boxed{f(t+T) = f(t)}$ for all real numbers t. The minimal value of T is called the period of f.

For example, $\sin(t)$ and $\cos(t)$ are periodic with period 2π.

If f is periodic with period T, then

$$\mathcal{L}(f)(s) = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

Consider

$$g_T(t) = f(t)\,u(t) - f_T(t)\,u_T(t)$$

On the one hand,

$$\mathcal{L}(g_T)(s) = \int_0^\infty g_T(t)e^{-st}\,dt$$

$$= \int_0^\infty (f(t)u(t) - f_T(t)u_T(t))e^{-st}\,dt$$

$$= \int_0^\infty f(t)u(t)e^{-st}\,dt - \int_0^\infty f_T(t)u_T(t)e^{-st}\,dt$$

$$= \int_0^\infty f(t)e^{-st}\,dt - \int_T^\infty f_T(t)e^{-st}\,dt$$

$$= \mathcal{L}(f)(s) - \int_T^\infty f_T(t)e^{-st}\,dt$$

$$= \mathcal{L}(f)(s) - \underbrace{\int_T^\infty f(t-T)e^{-st}\,dt}_{x = t - T}$$

$$\frac{dx}{dt} = 1$$

Using substitution, we get

$$= \mathcal{L}(f)(s) - \int_0^\infty f(x)e^{-s(x+T)}\,dt$$

$$= \mathcal{L}(f)(s) - e^{-sT}\mathcal{L}(f)(s)$$

$$= \mathcal{L}(f)(s)\left(1 - e^{-sT}\right)$$

However,

$$\mathcal{L}(g_T(t))$$

$$= \int_0^\infty \left(f(t)u(t) - f_T(t)u_T(t)\right)e^{-st}\,dt$$

$$= \int_0^\infty \left(f(t)u(t) - \underbrace{f(t-T)}_{=f(t)\text{ since } f \text{ is periodic of period } T.}u_T(t)\right)e^{-st}\,dt$$

$$= \int_0^\infty f(t)\left(u(t) - u(t-T)\right)e^{-st}\,dt$$

But

$$u(t) - u(t-T) = \begin{cases} 1, & 0 \le t < T \\ 0, & t \ge T \end{cases}$$

So the integral becomes

$\int_0^T f(t)e^{-st}\,dt$. Therefore

equating quantities,

$$\int_0^T f(t)e^{-st}\,dt = \mathcal{L}(f)(s)\left(1 - e^{-sT}\right)$$

so

$$\boxed{\mathcal{L}(f)(s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-sT}}}$$

Example 1: Solve

$$y'' + 3y' + 2y = \sin(t)$$

$$y(0) = 0, \quad y'(0) = 1.$$

Apply Laplace Transform to both sides:

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y)$$

$$= \mathcal{L}(\text{sine}) = \frac{1}{s^2+1}$$

We know

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$$
$$= s\mathcal{L}(y)$$
$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0)$$
$$= s^2\mathcal{L}(y) - 1$$

So the equation becomes

$$s^2\mathcal{L}(y) - 1 + 3s\mathcal{L}(y) + 2\mathcal{L}(y)$$
$$= \frac{1}{s^2+1}$$

So

$$\mathcal{L}(y)(s^2+3s+2) = \frac{1}{s^2+1} + 1,$$

so

$$\mathcal{L}(y) = \frac{1}{(s^2+1)(s^2+3s+2)} + \frac{1}{s^2+3s+2}$$

Partial Fractions:

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s+2)$$

$s = -1$ $\quad$ $s = -2$

$B = 1$ $\quad$ $A = -1$

$$\boxed{\frac{1}{s^2+3s+2} = \frac{-1}{s+2} + \frac{1}{s+1}}$$

$$\frac{1}{(s^2+1)(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{s^2+1}$$

$$1 = A(s+2)(s^2+1) + B(s+1)(s^2+1)$$
$$+ (Cs+D)(s+1)(s+2)$$

$\underline{s=-1}$

$1 = A(2)$, $\boxed{A = 1/2}$

$\underline{s=-2}$

$1 = B(-5)$, $\boxed{B = -1/5}$

Substitute back in:

$$1 = \frac{1}{2}(s+2)(s^2+1) - \frac{1}{5}(s+1)(s^2+1)$$
$$+ (Cs+D)(s+1)(s+2)$$

$s=0$

$$1 = 1 - \frac{1}{5} + D(2)$$

$\boxed{D = \frac{1}{10}}$ – plug in

$$1 = \frac{1}{2}(s+2)(s^2+1) - \frac{1}{5}(s+1)(s^2+1)$$
$$+ \left(Cs + \frac{1}{10}\right)(s+1)(s+2)$$

$s=1$

$$1 = 3 - \frac{4}{5} + \left(C + \frac{1}{10}\right)6$$

$\boxed{C = -3/10}$